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Extension of A Thick Infinite Plate With A Circular Hole

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1 Introduction.

The plane stress solution [1] is conventionally employed to estimate the stress concentration due to a circular hole in a plate that is uniaxially atretched at infinity. However, this solution is not a solution of the exact theory.[1] Nevertheless, we expect it to yield an accurate approximation if $\varepsilon = \frac{h}{R}$ is sufficiently "small".[2] We also expect that the accuracy will increase as $\varepsilon \to 0$.

The precise relationship between the plane stress and the exact theories is given in [3] for simply connected plates with "smooth" boundary curves. In addition, a boundary layer procedure[3] is outlined for obtaining increasingly accurate approximations to the solution of the exact theory. These approximations are given as "three-dimensional corrections" to the plane stress solution.

In this paper we extend the method of [3] to our stress concentration problem. Results are obtained in the form of a power series in ε. We give here only terms up to and including

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[1] We refer to the three-dimensional linear theory of elasticity for homogeneous and isotropic materials as the exact theory. There are some special cases for which the plane stress solution is a solution of the exact theory [2].

[2] Here, h is one half of the plate thickness and R is the radius of the hole.

[3] It is a generalization of the one given by Friedrichs [4] and Friedrichs and Dressler [5] in a study of the "bending" of plates. See also [6] and [7].

second order. Within this approximation we show that the plane stress theory yields extremely accurate, although non-conservative predictions of the maximum stress concentration for "small" but finite values of ε. This accuracy depends upon Poisson's ratio ν. For example, with $\nu = 1/3$ the error in the plane stress solution is less than 5% if $\varepsilon \leq .3$. For "larger" values of ν and ε, the error increases. More accurate approximations to the solution of the exact theory, which may be necessary for these values of ε, can be obtained by determining third and higher terms in the expansion.

Other approximate solutions of the exact theory for this problem are given by Sternberg and Sadowsky [8] using a modification of the Ritz method and Green [9] and Alblas [10] who employed infinite series expansions.[4] Our results for the stress concentration compare favorably with those of Alblas if $\varepsilon \leq \frac{1}{4}$, see Figs. 1 and 2. Agreement is especially good with his "asymptotic" solution which is closely related to our approximation method.

2. Formulation.

We introduce a cylindrical coordinate system r,θ,z. An infinite plate of thickness $2h$ with a circular hole of radius R is considered as a three-dimensional elastic body bounded by the planes (the "faces" of the plate) $z = \pm h$ and the cylindrical surface $r = R$. The origin of the coordinates is fixed at the center of the hole on the midplane of the plate, $z = 0$. The plate

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[4] See also Reissner [11].

is stretched at "infinity" by a constant tensile force T in a fixed direction which we take as the x-axis. The boundary of the hole, i.e. the edge, and the faces of the plate are free of forces.

If we introduce the dimensionless variables:

$$\xi = \frac{r-R}{R}\ , \quad \xi \geq 0\ ; \quad \zeta = \frac{z}{h}\ , \quad |\zeta| \leq 1\ ,$$

and the parameter,

$$\varepsilon = \frac{h}{R}\ ,$$

then the faces of the plate are given by $\zeta = \pm 1$ and the boundary of the hole by $\xi = 0$.

Considering the components of stress as functions of ξ, θ, ζ and employing an obvious notation, the stress formulation of the exact theory is given by:

Equilibrium Equations,

$$\sigma_{rz,\zeta} + \varepsilon[\sigma_{r,\xi} + (1+\xi)^{-1}(\sigma_{r\theta,\theta} + \sigma_r - \sigma_\theta)] = 0\ ,$$

$$(1) \qquad \sigma_{\theta z,\zeta} + \varepsilon[\sigma_{r\theta,\xi} + (1+\xi)^{-1}(\sigma_{\theta,\theta} + \sigma_{r\theta})] = 0\ ,$$

$$\sigma_{z,\zeta} + \varepsilon[\sigma_{rz,\xi} + (1+\xi)^{-1}(\sigma_{\theta z,\theta} + \sigma_{rz})] = 0\ ;$$

$$
(2) \quad
\begin{aligned}
&\sigma_{z,\zeta\zeta} + \Omega_{,\zeta\zeta} = -\varepsilon^2 \Delta\sigma_z , \\
&\sigma_{rz,\zeta\zeta} = -\varepsilon\Omega_{,\xi\zeta} + \varepsilon^2[-\Delta\sigma_{rz} + (1+\xi)^{-2}(\sigma_{rz} + 2\sigma_{\theta z,\theta})] , \\
&\sigma_{\theta z,\zeta\zeta} = -\varepsilon\Omega_{,\theta\zeta} + \varepsilon^2[-\Delta\sigma_{\theta z} + (1+\xi)^{-2}(\sigma_{\theta z} - 2\sigma_{rz,\theta})] , \\
&\sigma_{r,\zeta\zeta} = \varepsilon^2 A , \quad \sigma_{\theta,\zeta\zeta} = \varepsilon^2 B , \quad \sigma_{r\theta,\zeta\zeta} = \varepsilon^2 C .
\end{aligned}
$$

Here,

(3a) $A(\xi,\theta,\zeta;\varepsilon) = -\Delta\sigma_r - \Omega_{,\xi\xi} + 2(1+\xi)^{-2}(\sigma_r - \sigma_\theta + 2\sigma_{r\theta,\theta})$,

(3b) $B(\xi,\theta,\zeta;\varepsilon) = -\Delta\sigma_\theta - (1+\xi)^{-1}\Omega_{,\xi} + 2(1+\xi)^{-2}(-\Omega_{,\theta\theta} + 2\sigma_\theta - 2\sigma_r - 4\sigma_{r\theta,\theta})$,

(3c) $C(\xi,\theta,\zeta;\varepsilon) = -\Delta\sigma_{r\theta} - (1+\xi)^{-1}\Omega_{,\xi\theta} + (1+\xi)^{-2}(\Omega_{,\theta} + 4\sigma_{r\theta} + \sigma_{\theta,\theta} - \sigma_{r,\theta})$,

$$\Delta\sigma = \sigma_{,\xi\xi} + (1+\xi)^{-1}\sigma_{,\xi} + (1+\xi)^{-2}\sigma_{,\theta\theta} , \quad \Omega = \frac{1}{1+\nu}(\sigma_r + \sigma_\theta + \sigma_z) ,$$

ν is Poisson's ratio and a comma indicates partial differentiatior To complete the formulation we require appropriate boundary conditions. These are obtained by specifying the applied forces as:

(4a) $\sigma_{rz}(\xi,\theta,\pm 1) = \sigma_{\theta z}(\xi,\theta,\pm 1) = \sigma_z(\xi,\theta,\pm 1) = 0$;

(4b) $\sigma_{rz}(0,\theta,\zeta) = \sigma_r(0,\theta,\zeta) = \sigma_{r\theta}(0,\theta,\zeta) = 0$;

(4c) $\sigma_{rz}(\infty,\theta,\zeta) = 0$, $\sigma_r(\infty,\theta,\zeta) = T\cos^2\theta$, $\sigma_{r\theta}(\infty,\theta,\zeta) = -\frac{1}{2}T\sin 2\theta$,

where T is the constant tensile force at infinity in the x-direction.

Since we are concerned only with the extension of plates, it can be shown [5], without loss of generality, that σ_r, σ_θ, σ_z and $\sigma_{r\theta}$ are even functions of ζ while σ_{rz} and $\sigma_{\theta z}$ are odd functions of ζ. In the following sections we shall make frequent use of these properties without explicit reference.

3. The Interior Problems.

We assume that each stress component, indicated by the generic symbol $\sigma(\xi,\theta,\zeta;\varepsilon)$, can be asymptotically represented by a power series in ε:

$$(5) \qquad \sigma(\xi,\theta,\zeta;\varepsilon) \sim \sum_{n=0}^{\infty} \sigma^n(\xi,\theta,\zeta)\varepsilon^n \ .$$

Here σ^n are called the interior stress coefficients and we define $\sigma^n \equiv 0$ if $n < 0$. We assume that the σ^n possess the same even or odd property as σ. Substituting these expansions into (1-3) and (4a) and equating coefficients of like powers of ε yields a system of differential equations that are satisfied by the σ^n. The analysis of this system is elementary and similar to that outlined in [3]. Therefore, the calculations are not explicitly exhibited. Instead some of the results are listed below. For example, we can show that:

$$\left.\begin{array}{ll}
(6) & \sigma^n_{rz} = \sigma^n_{\theta z} = \sigma^n_z = 0 \ , \quad \Omega^n(\xi,\theta,\zeta) = \Omega^n(\xi,\theta) \ , \\[2ex]
(7) & \left\{\begin{array}{l}
\sigma^n_{r,\xi} + (1+\xi)^{-1}(\sigma^n_{r\theta,\theta} + \sigma^n_r - \sigma^n_\theta) = 0 \ , \\[1ex]
\sigma^n_{r\theta,\xi} + (1+\xi)^{-1}(\sigma^n_{\theta,\theta} + \sigma^n_{r\theta}) = 0 \ , \quad \Delta\Omega^n = 0 \ ;
\end{array}\right.
\end{array}\right\} \ n=0,1,\ldots$$

(8a) $\sigma_r^n = S_r^n(\xi,\theta),\ \ \sigma_\theta^n = S_\theta^n(\xi,\theta),\ \ \sigma_{r\theta}^n = S_{r\theta}^n(\xi,\theta), \qquad n = 0,1\ ;$

(8b)
$$\sigma_r^n = \frac{1}{2} A^{n-2}(\xi,\theta)\zeta^2 + S_r^n(\xi,\theta),\ \ \sigma_\theta^n = \frac{1}{2} B^{n-2}(\xi,\theta)\zeta^2 + S_\theta^n(\xi,\theta)\ ;$$
$$\sigma_{r\theta}^n = \frac{1}{2} C^{n-2}\zeta^2 + S_{r\theta}^n(\xi,\theta)\ , \qquad n = 2,3\ ,$$

where A^n, B^n and C^n are obtained from (3) using (5).

We define the m-th order interior problem (Problem I^m) as the boundary value problem that contains (6-8) with $n = m \leq 3$ and appropriate boundary conditions which are obtained in the following sections. Equations (6-8) with $n = 0$ are, in our notation, the stress relations and the differential equations of the classical theory of plane stress in polar coordinates [1].

4. Formulation of the Boundary Layer Problem.

The results given in the previous section are obtained without reference to the edge boundary conditions (4b). In fact, the expansions (5) cannot, in general, satisfy these conditions. If they did, it then follows from (8) and (3) that more boundary conditions then are appropriate for the solution of (7) are specified. Thus, if the series (5) represent the solution of the exact theory they do so in a region away from the edge which we call the "interior domain". The region of the plate adjacent to and including the edge where (5) may deviate rapidly from the solution of the exact theory is called the "boundary layer". To obtain expansions that may converge uniformly up to and including the edge we assume, as in [3]*, that the deviation of (5) from

* See also [6,7].

the exact solution occurs only in the direction normal to the edge, i.e. in the ξ direction. We then introduce the "stretched" boundary layer variable [12] η as:

$$\eta = \frac{\xi}{\varepsilon} \ . \tag{9}$$

Considering the η,θ,ζ coordinate system, boundary layer stresses indicated by the generic symbol $f(\eta,\theta,\zeta;\varepsilon)$ are defined as:

$$f(\eta,\theta,\zeta;\varepsilon) = \sigma(\xi,\theta,\zeta;\varepsilon) \ . \tag{10}$$

To determine approximations to the exact stress distribution near the edge of the hole and appropriate boundary conditions for the interior problems we assume that:

$$f(\eta,\theta,\zeta;\varepsilon) \sim \sum_{n=0}^{\infty} f^n(\eta,\theta,\zeta)\varepsilon^n \ . \tag{11}$$

The f^n are called the boundary layer stress coefficients and we set $f^n \equiv 0$ if $n < 0$. We further assume that each f^n possesses the same even or odd property as the corresponding f. Substituting (9-10) into the exact theory (1-4) and equating coefficients of like powers of ε we find from the coefficients of ε^n that the f^n satisfy:

$$\begin{aligned} & f^n_{rz,\zeta} + f^n_{r,\eta} + {\sum}'(f^i_{r\theta,\theta} + f^i_r - f^i_\theta) = 0 \ , \\ & f^n_{z,\zeta} + f^n_{rz,\eta} + {\sum}'(f^i_{\theta z,\theta} + f^i_{rz}) = 0 \ , \\ & f^n_{\theta z,\zeta} + f^n_{r\theta,\eta} + {\sum}'(f^i_{\theta,\theta} + 2f^i_{r\theta}) = 0 \ ; \end{aligned} \tag{12}$$

$$\nabla^2 f_r^n + \Gamma^n_{,\eta\eta} + {\sum}' f^i_{r,\eta} + {\sum}''\left[f^i_{r,\theta\theta} - 2(f^i_r - f^i_\theta + 2f^i_{r\theta,\theta})\right] = 0 ,$$

$$\nabla^2 f_z^n + \Gamma'^n_{,\zeta\zeta} + {\sum}' f^i_{z,\eta} + {\sum}'' f^i_{z,\theta\theta} = 0 ,$$

$$\nabla^2 f_\theta^n + {\sum}'(f^i_{\theta,\eta} + \Gamma^i_{,\eta}) + {\sum}''\left[2(f^i_r - f^i_\theta + 2f^i_{r\theta,\theta}) + f^i_{\theta,\theta\theta} + \Gamma^i_{,\theta\theta}\right] = 0 , \tag{13}$$

$$\nabla^2 f_{rz}^n + \Gamma^n_{,\eta\zeta} + {\sum}' f^i_{rz,\eta} + {\sum}''(f^i_{rz,\theta\theta} - f^i_{rz} - 2f^i_{\theta z,\theta}) = 0 ,$$

$$\nabla^2 f_{r\theta}^n + {\sum}'(f^i_{r\theta,\eta} + \Gamma^i_{,\eta\theta}) + {\sum}''\left[f^i_{r\theta,\theta\theta} + 2\frac{\partial}{\partial\theta}(f^i_r - f^i_\theta) - 4f^i_{r\theta} - \Gamma^i_{,\theta}\right] = 0 ,$$

$$\nabla^2 f_{\theta z}^n + {\sum}'(f^i_{\theta z,\eta} + \Gamma^i_{,\theta\zeta}) + {\sum}''\left[f^i_{\theta z,\theta\theta} - f^i_{\theta z} + 2f^i_{rz,\theta}\right] = 0 ;$$

$$f^n_{rz}(\eta,\theta,\pm 1) = f^n_z(\eta,\theta,\pm 1) = f^n_{\theta z}(\eta,\theta,\pm 1) = 0 ,$$

$$f^n_{rz}(0,\theta,\zeta) = f^n_r(0,\theta,\zeta) = f^n_{r\theta}(0,\theta,\zeta) = 0 , \tag{14}$$

$$f^n_{rz}(\infty,\theta,\zeta) = 0 , \qquad f^n_r(\infty,\theta,\zeta) = \begin{cases} T\cos^2\theta, & n = 0 \\ 0 , & n \neq 0 \end{cases} .$$

$$f^n_{r\theta}(\infty,\theta,\zeta) = \begin{cases} -T/2 \sin 2\theta, & n = 0 \\ 0 , & n \neq 0 \end{cases} .$$

Here,

$$\Gamma^n = \frac{1}{1+\nu}(f^n_r + f^n_\theta + f^n_z) , \qquad \nabla^2 \equiv \frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\zeta^2} ,$$

$${\sum}' A^i = \sum_{i+j+1=n}\sum (-1)^j \eta^j A^i , \qquad {\sum}'' A^i = \sum_{i+j+2=n}\sum (-1)^j (j+1)\eta^j A^i .$$

In addition we require the boundary layer stresses to approach or "match" the interior stresses as $\varepsilon \longrightarrow 0$. Specifically, we assume that each $\sigma^n(\xi,\theta,\zeta)$ has, near $\xi = 0$, a Taylor series expansion in ξ:

$$\sigma^n(\xi,\theta,\zeta) = \sum_{m=0}^{\infty} s_m^n(\theta,\zeta)\xi^m ,$$

where

(15a) $$s_m^n = \frac{1}{m!}\frac{\partial^m \sigma^n(0,\theta,\zeta)}{\partial\xi^m} , \qquad n = 0,1,\dots .$$

Therefore from (5) and (9) we have, in some region about $\xi = 0$,

(16) $$\sigma(\xi,\theta,\zeta;\varepsilon) \sim \sum_{n=0}^{\infty} \overset{*}{\sigma}{}^n(\eta,\theta,\zeta)\varepsilon^n ,$$

where

(15b) $$\overset{*}{\sigma}{}^n(\eta,\theta,\zeta) = \sum_{m=0}^{n} s_m^{n-m}(\theta,\zeta)\eta^m$$

are the interior coefficients near $\xi = 0$ as functions η, θ and ζ. For each n we define the "reduced boundary layer stress coefficients", $F^n(\eta,\theta,\zeta)$ as:

(17) $$F^n(\eta,\theta,\zeta) \equiv f^n(\eta,\theta,\zeta) - \overset{*}{\sigma}{}^n(\eta,\theta,\zeta) .$$

The "matching condition" or the asymptotic form for the f^n is obtained from (9), (10) and (16) by associating each f^n with the corresponding $\overset{*}{\sigma}{}^n$ as $\varepsilon \to 0$. Using (9) and (17) we write this condition as:†

(15c) $$\lim_{\eta\to\infty} F^n(\eta,\theta,\zeta) = 0 , \qquad n = 0,1,\dots .$$

An analysis, similar to the preceding, applied to the boundary at "infinity" yields a corresponding formulation of that boundary layer problem. We do not exhibit this formulation.

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† In obtaining (15c) we assume that the terms in f^n which vanish as $\eta \to \infty$ do so faster than any negative power of η.

5. Analysis of the Boundary Layer Problems.

For each n Eqs. (12-15) and (17) separate into two distinct systems which we call Problem P^n and Problem T^n. Problem P^n is concerned with the coefficients f_r^n, f_z^n, f_θ^n and f_{rz}^n and involves the first two of (12), the first four of (13), (15) and (17) for these coefficients and the first two of each of (14). Problem T^n is concerned with the remaining two coefficients $f_{r\theta}^n$ and $f_{\theta z}^n$ and the remaining equations in (12-15) and (17).

We shall associate with P^n a "stress function", $\phi^n(\eta,\theta,\zeta)$, which may be the solution of the following boundary value problem on the semi-infinite strip, $|\zeta| \leq 1$, $\eta \geq 0$, and fixed θ:

$$\nabla^4\phi^n = 0 \; ;$$

(18)
$$\phi^n_{,\eta\eta}(\eta,\theta,\pm 1) = \phi^n_{,\eta\zeta}(\eta,\theta,\pm 1) = 0; \qquad \lim_{\eta\to\infty} [\phi^n_{,\zeta\zeta}, \phi^n_{,\eta\zeta}] = 0 \; ;$$
$$\phi^n_{,\zeta\zeta}(0,\theta,\zeta) = \beta^n(\theta,\zeta) \; , \qquad \phi^n_{,\eta\zeta}(0,\theta,\zeta) = 0 \; ,$$

where $\beta^n(\theta,\zeta) = \beta^n(\theta,-\zeta)$. It can be shown by elementary means that if $\phi^n_{,\zeta\zeta}$, $\phi^n_{,\eta\zeta}$ and $\phi^n_{,\zeta}$ are uniformly continuous functions of ζ and if ϕ^n, $\phi^n_{,\eta}$ and $\phi^n_{,\zeta}$ are single-valued functions then,

(19)
$$\int_{-1}^{1} \beta^n(\theta,\zeta)d\zeta = 0 \qquad \text{for all } \theta \; .$$

Employing Problem I^0, we can show that a solution of P^0 is given by

(20a)
$$F_r^0 = \phi^0_{,\zeta\zeta}, \quad F_z^0 = \phi^0_{,\eta\eta}, \quad F_{rz}^0 = -\phi^0_{,\eta\zeta}, \quad F_\theta^0 = \nu\nabla^2\phi^0$$

if ϕ^0 is the solution of (18) with n = 0 and

(20b) $$\beta^0(\theta,\zeta) = -S_r^0(0,\theta) .$$

The first boundary condition on $\xi = 0$ for Problem I^0 is obtained from (20b) and (19) as,

(21) $$S_r^0(0,\theta) = 0 .$$

Using Problem I^0 it follows that

(22a) $$F_{r\theta}^0 = -\psi^0_{,\zeta} \quad , \qquad F_{\theta z}^0 = \psi^0_{,\eta}$$

is a solution of Problem T^0 if $\psi^0(\eta,\theta,\zeta)$ is the solution of,

(23) $$\nabla^2\psi^n = 0 ,$$
$$\psi^n_{,\eta}(\eta,\theta,\pm1) = 0, \quad \lim_{\eta\to\infty} [\psi^n_{,\zeta}, \psi^n_{,\eta}] = 0, \quad \psi^n_{,\zeta}(0,\theta,\zeta) = g^n(\theta,\zeta)$$

with n = 0 where,

(22b) $$g^0(\theta,\zeta) = -S_{r\theta}^0(0,\theta) .$$

Single valuedness of the solution of (23) yields from (22b) the second boundary condition on $\xi = 0$ for I^0 as,

(24) $$S_{r\theta}^0(0,\theta) = 0 .$$

From (18) and (20-24) it follows that

(25) $$F^0(\eta,\theta,\zeta) \equiv 0 .$$

In a similar manner it may be shown from an analysis of Problems P^1 and T^1 that

(26) $$F^1(\eta,\theta,\zeta) \equiv 0$$

and that the boundary conditions on $\xi = 0$ for Problem I^1 are given by,

(27) $$S_r^1(0,\theta) = S_{r\theta}^1(0,\theta) = 0 \ .$$

A solution of Problem P^2 is obtained from (18) with $n = 2$ and

(28a) $$\beta^2(\theta,\zeta) = -S_r^2(0,\theta) - \frac{1}{2} A^0(0,\theta)\zeta^2$$

if

(28b) $$F_r^2 = \phi^2_{,\zeta\zeta}, \quad F_z^2 = \phi^2_{,\eta\eta}, \quad F_{rz}^2 = -\phi^2_{,\eta\zeta}, \quad F_\theta^2 = \nu\nabla^2\phi^2 \ .$$

Equations (19) and (28a) yield the first boundary condition for Problem I^2 as,

(29) $$S_r^2(0,\theta) = -\frac{1}{6} A^0(0,\theta)$$

where A^0 is obtained from (5) and (3a). Similarly we obtain a solution of Problem T^2 by setting

$$F_{r\theta}^2 = -\psi^2_{,\zeta} \ , \quad F_{\theta z}^2 = \psi^2_{,\eta}$$

where $\psi^2(\eta,\theta,\zeta)$ is a solution of (23) with $n = 2$ and

$$g^2(\theta,\zeta) = S_{r\theta}^2(0,\theta) + \frac{1}{2} C^0(0,\theta)\zeta^2 \ .$$

Single valuedness of the solution gives the second boundary condition for Problem I^2 as,

$$S^2_{r\theta}(0,\theta) = -\frac{1}{6}\, C^0(0,\theta)\ . \tag{30}$$

More accurate approximations of the stresses near the edge can be obtained by examining Problems P^n and T^n for $n \geq 3$.

A corresponding analysis of the boundary layer at "infinity' yields the appropriate boundary conditions at infinity for the interior problems. For example, we can show that,

$$S^0_r(\infty,\theta) = \frac{T}{2}(1+\cos 2\theta)\ , \qquad S^0_{r\theta}(\infty,\theta) = -\frac{T}{2}\sin 2\theta\ , \tag{31a}$$

$$S^n_r(\infty,\theta) = S^n_{r\theta}(\infty,\theta) = 0 \qquad\qquad \text{if} \quad n = 1,2\ . \tag{31b}$$

6. <u>Solution of the Interior and Boundary Layer Problems.</u>

Problem I^0 (the plane stress theory) which consists of the differential equations (7) and (8a) with n = 0 and the boundary conditions (21), (24) and (31a) has the solution [1]:

$$\frac{2}{T}\sigma^0_r = \frac{2}{T}S^0_r = 1 - \frac{1}{(1+\xi)^2} + \left[1 + \frac{3}{(1+\xi)^4} - \frac{4}{(1+\xi)^2}\right]\cos 2\theta\ ,$$

$$\frac{2}{T}\sigma^0_\theta = \frac{2}{T}S^0_\theta = 1 + \frac{1}{(1+\xi)^2} - \left[1 + \frac{3}{(1+\xi)^4}\right]\cos 2\theta\ , \tag{32}$$

$$\frac{2}{T}\sigma^0_{r\theta} = \frac{2}{T}S^0_{r\theta} = -\left[1 + \frac{2}{(1+\xi)^2} - \frac{3}{(1+\xi)^4}\right]\sin 2\theta\ ,$$

$$\sigma^0_{rz} = \sigma^0_{\theta z} = \sigma^0_z = 0\ .$$

Since the differential equations (7) and the boundary conditions (27) and (31b) are homogeneous we have for the solution of Problem I^1:

$$\sigma^1 \equiv 0 \ . \tag{33}$$

Employing (32) and (3) we obtain the solution of Problem I^2, which is given by (7) and (8b) with n = 2 and (29), (30) and (31b), as

$$\begin{aligned} &\sigma_r^2 = -\sigma_\theta^2 = G \cos 2\theta, \quad \sigma_{r\theta}^2 = G \sin 2\theta, \\ &\sigma_{rz}^2 = \sigma_{\theta z}^2 = \sigma_z^2 = 0 \ , \end{aligned} \tag{34a}$$

where

$$G(\xi,\zeta) = \frac{2\nu T}{(1+\nu)} \frac{(1 - 3\zeta^2)}{(1+\xi)^4} \ . \tag{34b}$$

To obtain a solution of Problem P^2 we must solve the boundary value problem (18) with n = 2 where β^2 is given by (28a), (29), (3a) and (32). Since an exact solution of this problem is unknown we employ the "approximate" solution given by Horvay [13]:

$$\begin{aligned} &F_r^2 = \frac{H}{3} e^{-a\eta}(\cos b\eta + \frac{a}{b} \sin b\eta)(3\zeta^2 - 1) \cos 2\theta \ , \\ &F_z^2 = H(\frac{a^2+b^2}{12}) \ e^{-a\eta} \ (-\cos b\eta + \frac{a}{b} \sin b\eta)(1-\zeta^2)^2 \cos 2\theta \ , \\ &F_{rz}^2 = H(\frac{a^2+b^2}{3b}) \ e^{-a\eta} \sin b\eta \ (\zeta - \zeta^3) \cos 2\theta \ , \\ &F_\theta^2 = \nu(F_r^2 + F_z^2) \ , \end{aligned} \tag{35a}$$

where

(35b) $$H = \frac{6\nu T}{1+\nu} \quad , \quad a = 2.075 \quad , \quad b = 1.143 \ .$$

An integral representation can be given for the solution of Problem T^2 since Green's function for the boundary value problem (23) is known [14]. However, we prefer to use the infinite series representation which yields for the solution of T^2:

(36a)
$$F^2_{r\theta} = \sum_{n=1}^{\infty} K_n \cos n\pi\zeta \; e^{-n\pi\zeta} \sin 2\theta \quad ,$$
$$F^2_{\theta z} = \sum_{n=1}^{\infty} K_n \sin n\pi\zeta \; e^{-n\pi\zeta} \sin 2\theta \quad ,$$

where,

(36b) $$K_n = \frac{4H}{\pi^2} \frac{(-1)^n}{n^2} \ .$$

As in [3] we define the stresses, $\sigma^{(N)}$, of the <u>N-th approximation to the exact theory</u> as:

(37) $$\sigma^{(N)}(\xi,\theta,\zeta;\varepsilon) = \sum_{n=0}^{N} [\sigma^n(\xi,\theta,\zeta) + F^n(\xi/\varepsilon,\theta,\zeta)]\varepsilon^n \ .$$

It follows from this definition, (25), (26) and (33) that $\sigma^{(0)}$ and $\sigma^{(1)}$ coincide with the plane stress solution. For the second approximation to the exact theory we obtain from (37):

$$
(38)\quad
\begin{aligned}
\sigma_r^{(2)}(\xi,\theta,\zeta;\varepsilon) &= S_r^0(\xi,\theta)+[\sigma_r^2(\xi,\theta,\zeta)+F_r^2(\xi/\varepsilon,\theta,\zeta)]\varepsilon^2 ,\\
\sigma_\theta^{(2)}(\xi,\theta,\zeta;\varepsilon) &= S_\theta^0(\xi,\theta)+[\sigma_\theta^2(\xi,\theta,\zeta)+F^2(\xi/\varepsilon,\theta,\zeta)]\varepsilon^2 ,\\
\tau_{r\theta}^{(2)}(\xi,\theta,\zeta;\varepsilon) &= S_{r\theta}^0(\xi,\theta)+[\sigma_{r\theta}^2(\xi,\theta,\zeta)+F_{r\theta}^2(\xi/\varepsilon,\theta,\zeta)]\varepsilon^2 ,\\
\sigma_{rz}^{(2)}(\xi,\theta,\zeta;\varepsilon) &= F_{rz}^2(\xi/\varepsilon,\theta,\zeta)\varepsilon^2 ,\\
\sigma_{\theta z}^{(2)}(\xi,\theta,\zeta;\varepsilon) &= F_{\theta z}^2(\xi/\varepsilon,\theta,\zeta)\varepsilon^2 ,\\
\sigma_z^{(2)}(\xi,\theta,\zeta;\varepsilon) &= F_z^2(\xi/\varepsilon,\theta,\zeta)\varepsilon^2 .
\end{aligned}
$$

Here S^0, σ^2 and F^2 are given in (32) and (34-36).

7. Presentation of Results.

We define the quantity:

$$
(39)\qquad \hat{\sigma}_\theta^{(2)}(\xi,\zeta;\varepsilon) = \frac{\dfrac{\sigma_\theta^{(2)}(\xi,\theta,\zeta;\varepsilon)}{T} - \dfrac{1}{2}\left[1+\dfrac{1}{(1+\xi)^2}\right]}{\cos 2\theta} .
$$

In Fig. 1, $\hat{\sigma}_\theta^{(2)}(0,0;\varepsilon)$ is illustrated as a function of ε with varying Poisson's ratio. For $\varepsilon = 0$ we obtain the plane stress result which gives a stress-concentration factor of two for $\hat{\sigma}_\theta^{(2)}$. Although the stress-concentration factor increases with ε, for "small" ε, it is only a small percentage of the plane stress solution. For example, with $\varepsilon = .2$ and $\nu = 1/4$ our results indicate only a 1.25 % increase over the plane stress result. Thus, for "small" ε the plane stress solution apparently yields

accurate but unconservative results. The circles in Fig. 1 represent the results of Alblas [10] ($\nu = 1/4$) obtained from a formal infinite series solution of the exact theory. The dotted curve gives the "asymptotic solution" of Alblas.

Figure 2 reveals the behavior of $\hat{\sigma}_\theta^{(2)}(0,\pm 1;\varepsilon)$ as a function of ε. The curve obtained from the "asymptotic solution" of Alblas ($\nu = 1/4$ and $\varepsilon \leq 1/4$) coincides with our curve. In Fig. 3 the variation through the thickness of $\hat{\sigma}^{(2)}$ is indicated at the edge of the hole for $\nu = 1/4$ and varying ε. Since $\hat{\sigma}_\theta^{(2)}$ assumes its maximum on the middle surface, these results are of some importance in experiments where measurements are taken on the faces of the plate.

In the remaining three figures we illustrate "boundary layer behaviors" for some of the stresses. In Fig. 4 the variation with ξ of the middle surface values of $\hat{\sigma}_\theta^{(2)}$ is given for $\nu = 1/4$ and two values of ε. The ξ-variation of $\hat{\sigma}_\theta^{(2)}$ is relatively insensitive to changes in ε especially for $\varepsilon > .2$. Using the plane stress solution we define $\hat{\sigma}_\theta^{(0)}$ as

$$(40) \qquad \hat{\sigma}_\theta^{(0)} = \frac{\frac{S_\theta^0}{T} - \frac{1}{2}\left[1 + \frac{1}{(1+\xi)^2}\right]}{\cos 2\theta} .$$

This quantity is in close agreement with $\hat{\sigma}_\theta^{(2)}(\xi,0;\varepsilon)$ for $\varepsilon = 1/10$ and is not shown in Fig. 4.

Figure 5 shows the ξ-variation of the scaled thickness shear stress,

$$(41) \qquad \hat{\sigma}^{(2)}_{z\theta}(\xi,\zeta;\varepsilon) = \left(\frac{100\pi^2(1+\nu)}{24\nu T \sin 2\theta} \right) \sigma^{(2)}_{z\theta}(\xi,\theta,\zeta;\varepsilon)$$

for $\zeta = 1/2$. We observe that for $\varepsilon = 1/4$, $\hat{\sigma}^{(2)}_{z\theta}$ at $\xi = .1$ is only 30 % of its maximum value, while at $\xi = .2$, $\hat{\sigma}^{(2)}_{z\theta}$ is 8.5 % of its maximum value. If we define the "boundary layer thickness", ξ^*, as that value of ξ at which $\hat{\sigma}^{(2)}_{z\theta}$ is a small percentage, say 5 %, of its maximum value then the results illustrated in Fig. 5 give

$$\xi^* \approx \varepsilon .$$

In the remaining graph, Fig. 6, the ξ variation of $\hat{\sigma}^{(2)}_{r\theta}$ is given on the face of the plate for $\nu = 1/4$. Here $\hat{\sigma}^{(2)}_{r\theta}$ is defined as,

$$(42) \quad \hat{\sigma}^{(2)}_{r\theta}(\xi,\zeta;\varepsilon) = 125 \left\{ \frac{2\sigma^{(2)}_{r\theta}(\xi,\theta,\zeta;\varepsilon)}{T \sin 2\theta} + \left[1 + \frac{2}{(1+\xi)^2} - \frac{3}{(1+\xi)^4} \right] \right\} .$$

For the plane stress solution, $\varepsilon = 0$, the corresponding $\hat{\sigma}^{(2)}_{r\theta}$ coincides with the ξ axis.

We wish to emphasize that more accurate approximations to the exact solution can be obtained by extending our calculations to terms of order three and greater. Additional accuracy in our solution (38) could be obtained if the exact solution to Problem P^2, rather than (35), were available. Since (36) provides the exact solution to Problem T^2, the stresses $\sigma^{(2)}_{r\theta}$ and $\sigma^{(2)}_{\theta z}$ are most likely given with greater precision than the other stresses in (38).

(44) [illegible]

that c = 1/3, the maximum [illegible] is [illegible] only [illegible] of its maximum value, while at [illegible] is [illegible] of its maximum value. If we include the [illegible] we find [illegible] is a small percentage [illegible] of its maximum value [illegible] discussed in [illegible].

§ 9.

In the [illegible] of [illegible] is given [illegible] is defined as

(45) [illegible]

For the plane stress solution [illegible] coincides with the [illegible].

We might [illegible] approximations to the exact elastic [illegible] in terms of [illegible] equation (45) would be [illegible] (24) [illegible] (25) [illegible] the stresses [illegible] of (45).

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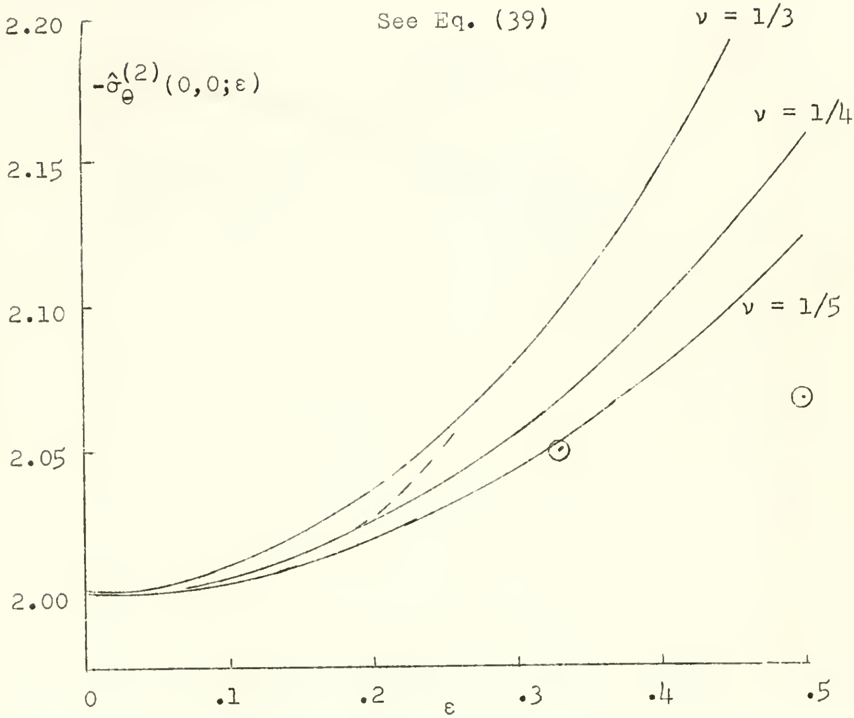


Figure 1: Variation with ε and ν of the middle plane values of $\hat{\sigma}_\theta^{(2)}$ at the edge of the hole.

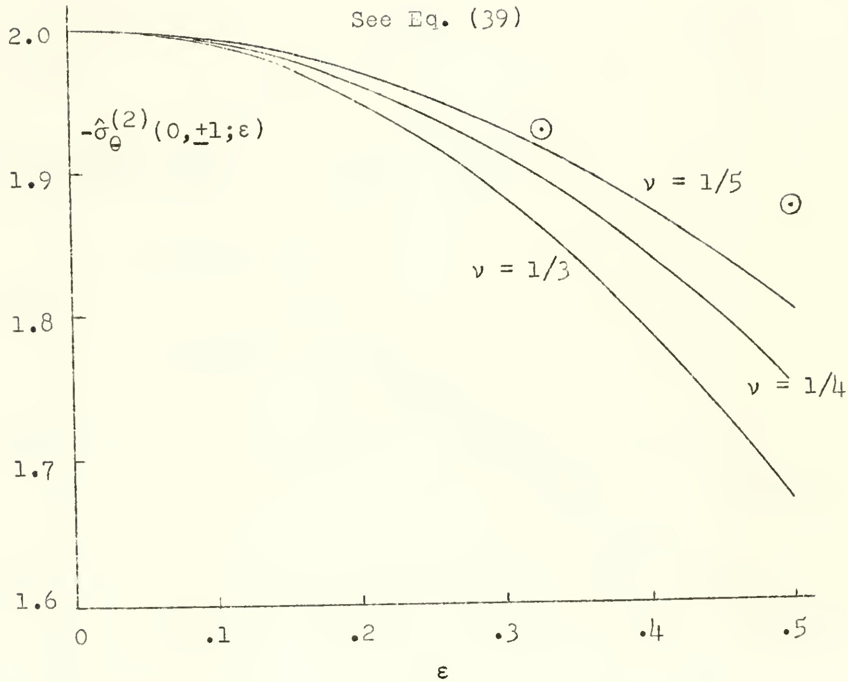


Figure 2: Variation with ε and ν of the face plane values of $\hat{\sigma}_\theta^{(2)}$ at the edge of the hole.

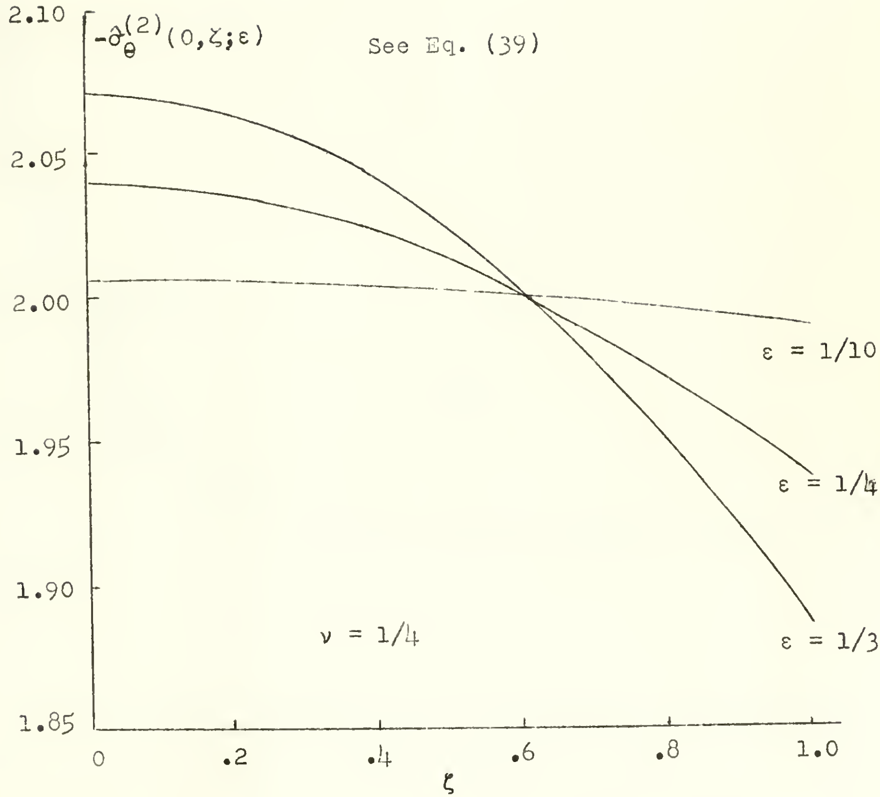
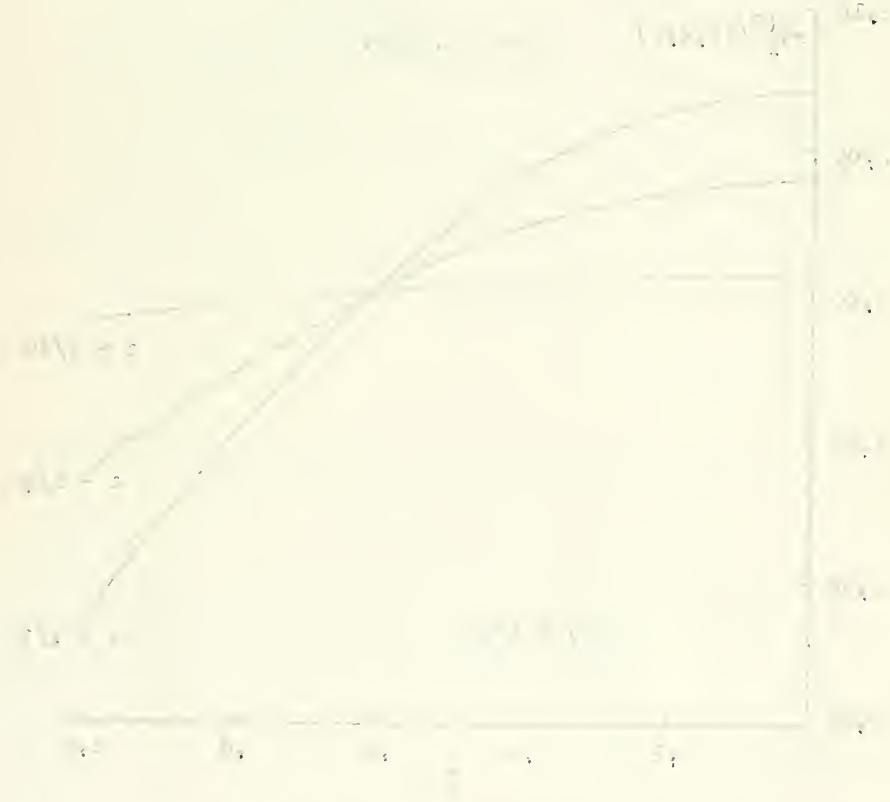


Figure 3: Variation through the thickness of $\hat{\sigma}_{\theta}^{(2)}$ at the edge of the hole for $\nu = 1/4$.



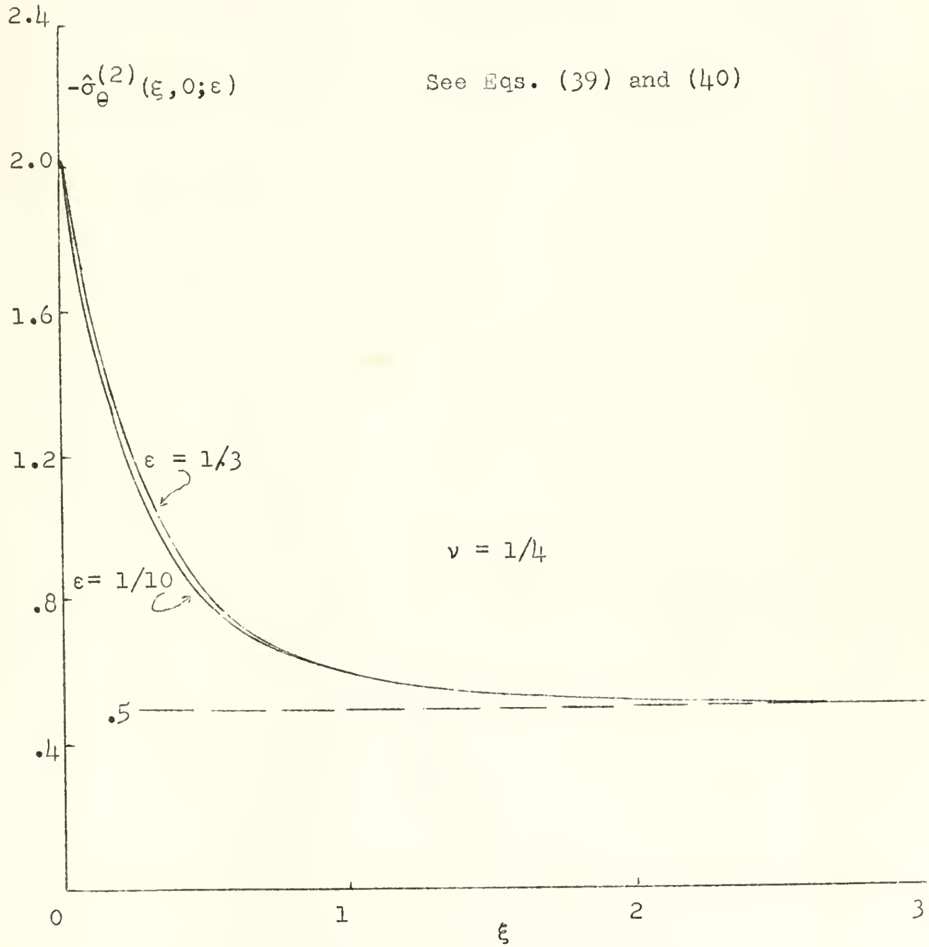
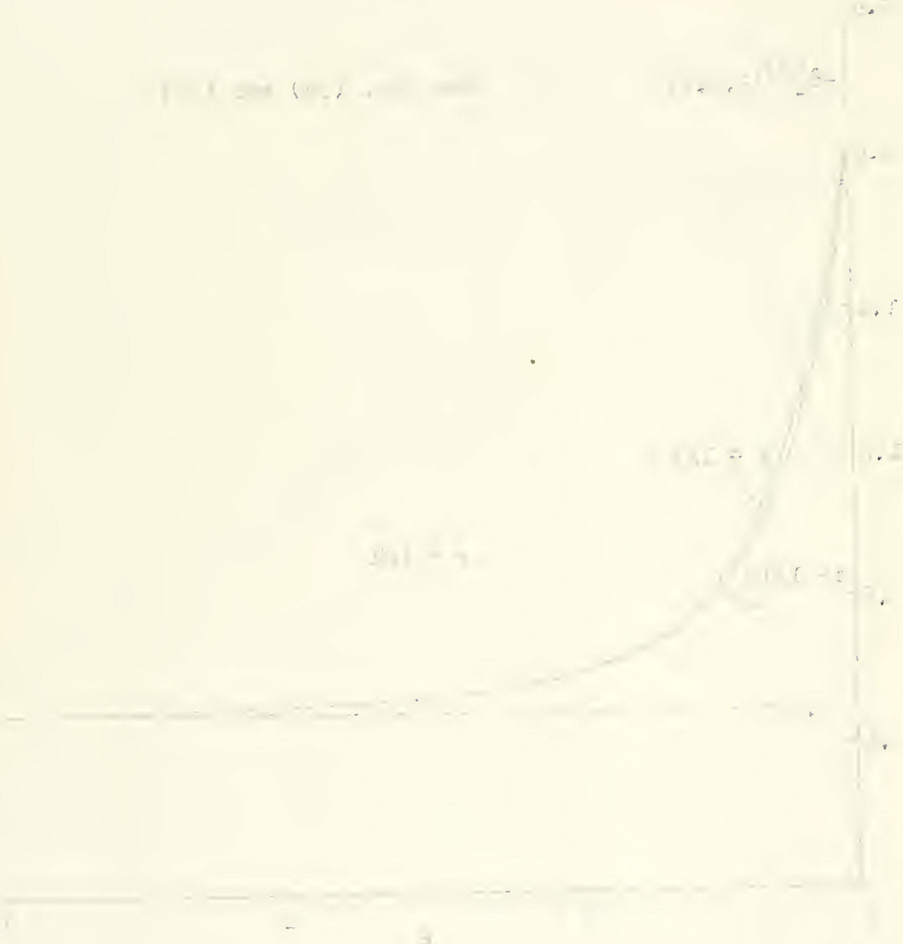


Figure 4: Variation with ξ of the middle plane values of $\hat{\sigma}_\theta^{(2)}$ for $\nu = 1/4$.



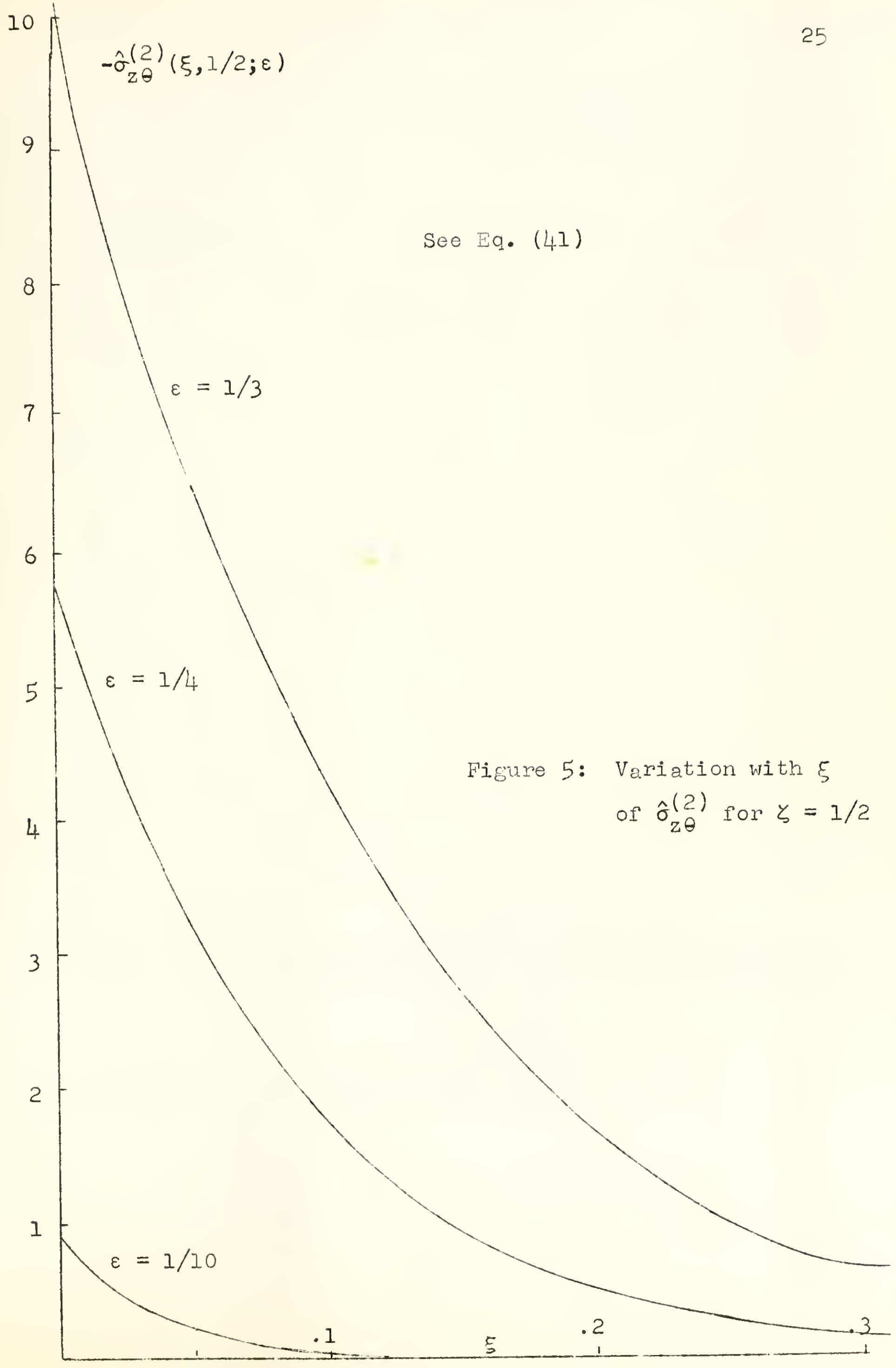


Figure 5: Variation with ξ of $\hat{\sigma}_{z\theta}^{(2)}$ for $\zeta = 1/2$

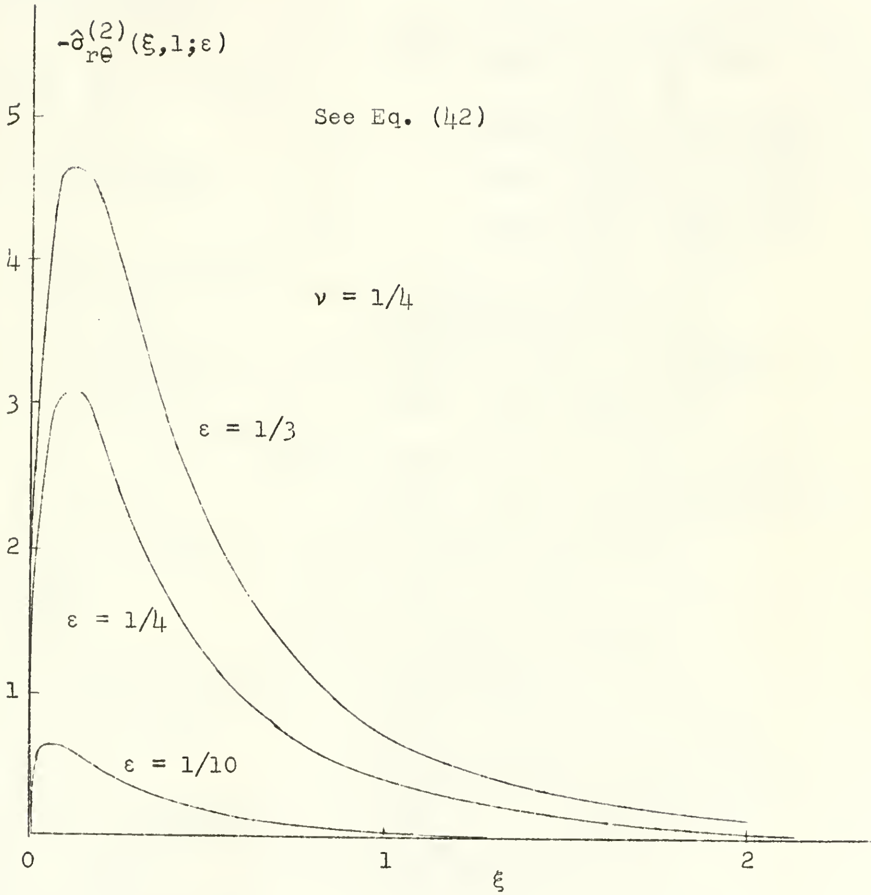


Figure 6: Variation with ξ of $\hat{\sigma}^{(2)}_{r\theta}$ for $\zeta = 1$.

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Cincinnati 15, Ohio (1)

Dr. F. Lane
General Applied Science Labs.
Stewart and Merrick Avenues
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Commander Edward Leonard
Asst. Navy Representative
MIT Lincoln Lab.
Lexington 73, Massachusetts (1)

DATE DUE	BORROWER'S NAME	ROOM NUMBER
JAN 18 '68	~~[illegible]~~	
~~N4SILL~~	~~Suny Buf~~	~~7/30/70~~
ILL	~~PINY~~	~~due 11/16/70~~